

Iterating evolutes of spatial polygons and of spatial curves

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1 Introduction

The evolute of a smooth plane curve is the locus of its centers of curvature or, equivalently, the envelope of the family of its normal lines. The construction of the evolute can be iterated. In our recent paper [1], we studied such iterations; we also investigated discrete versions of this problem where smooth curves are replaced by polygons.

In the present paper we study higher dimensional analogs of this problem, the iterations of the evolutes of polygons in \mathbb{R}^m and of closed smooth curves (possibly, with cusps) in \mathbb{R}^3 . Our investigation consists of two parts, concerning polygons and curves, respectively.

The first part of the paper concerns polygons in Euclidean spaces of arbitrary dimensions.

For an n -gon in \mathbb{R}^m with $n \geq m + 2$, we define the evolute as the n -gon whose vertices are the centers of the spheres through the $(m + 1)$ -tuples of its consecutive vertices. (There exists an alternative definition of an evolute of an n -gon as an n -gon whose vertices are the centers of spheres tangent to the $(m + 1)$ -tuples of consecutive sides; we briefly consider this variant but do not study it in any detail.)

A polygon \mathbf{P} is an involute of a polygon \mathbf{Q} if \mathbf{Q} is the evolute of \mathbf{P} . The existence and uniqueness of an involute of a generic n -gon in \mathbb{R}^m depends on

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n and m (and, of course, on which of the two definitions of the evolute is chosen). In Proposition 2.3, we give a complete answer to this question.

For an n -gon, its tangent indicatrix (the image of the tangent Gauss map) is a n -gon in S^{m-1} or, equivalently, a cyclically ordered sequence of n unit vectors in \mathbb{R}^m . Given a spherical n -gon \mathbf{v} , the space $\mathcal{P}_{\mathbf{v}}$ of the spatial n -gons with the respective directions of the sides, considered modulo parallel translations, is an $(n - m)$ -dimensional vector space, with the signed side lengths as coordinates.

The sides of the evolutes of the polygons in $\mathcal{P}_{\mathbf{v}}$ also have fixed directions, and the respective spherical polygon \mathbf{u} is spherically dual to \mathbf{v} . The sides of the second evolute are parallel to the sides of the original polygon.

The evolute map $\mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ is a linear map of $(n - m)$ -dimensional vector spaces, depending on \mathbf{v} (Theorem 1). Generically, this map has full rank if $n - m$ is even, and has a 1-dimensional kernel if $n - m$ is odd. The second evolute map is a linear self-map of $\mathcal{P}_{\mathbf{v}}$.

For a generic spherical n -gon \mathbf{v} , we define a non-degenerate pairing between the spaces $\mathcal{P}_{\mathbf{v}}$ and $\mathcal{P}_{\mathbf{u}}$ where $\mathbf{u} = \mathbf{v}^*$. Thus $\mathcal{P}_{\mathbf{v}^*} = (\mathcal{P}_{\mathbf{v}})^*$. The evolute map $\mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}^*}$ is anti-self-adjoint.

The case of “small-gons”, that is, $(m + 2)$ - and $(m + 3)$ -gons in \mathbb{R}^m , is special. We prove that the second evolute of an $(m + 2)$ -gon is homothetic to the original polygon (Theorem 2) and that the first and third evolutes of an $(m + 3)$ -gon are homothetic (Theorem 3). The first of these results is not new: it was proved by E. Tsukerman [11] by a different method. The second result was known in dimension 2: it was conjectured by B. Grünbaum in [7, 8] and proved in our paper [1].

In general, if the maximum module eigenvalue of the second evolute transformation is real, then the whole sequence of multiple evolutes of a polygon, considered up to parallel translations and scaling, is asymptotically 2-periodic (the limit coefficient of the homothety equals the maximum module eigenvalue; in particular, it may be negative). In Figures 1, 2 and 3, we show examples of such behavior for heptagons.

Our last result on polygons concerns the spectrum of the second evolute map. Theorem 4 states that every non-zero eigenvalue of this map has even multiplicity, generically, multiplicity 2. More precisely, the matrix of the restriction of the second evolute transformation to the image of the evolute transformation is conjugated to the block diagonal matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}.$$

The main ingredient of the proof is a construction of a linear symplectic structure on the space $\mathcal{P}_{\mathbf{v}}$, when $n - m$ is even, or on its quotient by the kernel of the evolute map, when $n - m$ is odd. The second evolute map is skew-Hamiltonian with respect to this symplectic structure.

Theorem 4 is our strongest result on polygons; it provides alternative proofs of Theorems 2 and 3.

In the second part, we consider curves in \mathbb{R}^3 with non-vanishing curvature and non-vanishing torsion (the former is a general position property, whereas the latter is not). Our curves may have cusps, such as the curve (t^2, t^3, t^4) at the origin. Each smooth piece of the curve is oriented, and the orientations agree at cusps (defining a coorientation of the curve). The number of cusps is necessarily even. The tangent Gauss image of such a curve, its tangent indicatrix, is an immersed closed curve on the unit sphere.

The osculating sphere of a spatial curve passes through a quadruple of infinitesimally close points of the curve. The *evolute* of a curve is the locus of the centers of its osculating spheres (this evolute is also called the evolute of the 2nd kind in [3] and the focal curve in [12]). Equivalently, the evolute is the enveloping curve of the family of the normal planes of a spatial curve, and it is called the edge of regression of the polar developable in [10]. See [5, 12] for a study of evolutes of spatial curves.

The space of curves in \mathbb{R}^3 modulo parallel translations fibers over the space of spherical curves, their tangent indicatrices. The space of curves \mathcal{C}_γ with a fixed tangent indicatrix $\gamma \subset S^2$ is an infinite-dimensional vector space. One of our results is that the tangent indicatrix $\bar{\gamma}$ of the evolute of a spatial curve is spherically dual to γ , the indicatrix of the original curve, and the evolute map $\mathcal{C}_\gamma \rightarrow \mathcal{C}_{\bar{\gamma}}$ is linear (see Theorem 5).

We prove continuous analogs of some of the results obtained for polygons. In particular, for a generic tangent indicatrix γ , we construct a non-degenerate pairing between the spaces \mathcal{C}_γ and $\mathcal{C}_{\bar{\gamma}}$, and we prove that the evolute map $\mathcal{C}_\gamma \rightarrow \mathcal{C}_{\bar{\gamma}}$ is an anti-self-adjoint linear bijection.

The second evolute of a spatial curve has the same, up to a central symmetry, tangent indicatrix as the original curve, and one wonders whether there exist curves which are homothetic to their second evolutes. In the plane, the curves with this property are the classical hypocycloids, see [1], Corollary 2.8.

We construct a family of spatial curves that are homothetic to their second evolutes. The tangent indicatrices of these curves are circles on the unit sphere. These curves may be regarded as spatial analogs of hypocycloids,

and indeed, they look somewhat like the classical hypocycloids, see Figure 4. We do not know whether there are other spatial curves homothetic to their second evolutes.

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2 Polygons

We consider a closed n -gon $\mathbf{P} = (P_1, P_3, \dots, P_{2n-1})$ in \mathbb{R}^m (the numeration of vertices is cyclic); we assume that $n \geq m + 2$ and that no m consecutive sides of \mathbf{P} are parallel to the same $(m - 1)$ -dimensional plane.

2.1 Evolutes: definitions

There are two natural notions of the evolute of the polygon \mathbf{P} ; following the terminology of [1], we call them \mathcal{P} -evolute and \mathcal{A} -evolute.

The \mathcal{P} -evolute \mathbf{Q} (resp., \mathcal{A} -evolute \mathbf{R}) of \mathbf{P} is the n -gon whose (cyclically ordered) vertices are centers of the spheres passing through (resp., tangent to) the $(m + 1)$ -tuples of (cyclically) consecutive vertices (resp., sides) of \mathbf{P} . The most convenient way of numeration of the vertices of the evolutes is as follows. The vertex Q_i of \mathbf{Q} is the center of the sphere passing through the points

$$P_{i-m}, P_{i-m+2}, P_{i-m+4}, \dots, P_{i+m-4}, P_{i+m-2}, P_{i+m},$$

and the vertex R_i of \mathbf{R} is the center of the sphere tangent to the sides

$$\overline{P_{i-m-1}P_{i-m+1}}, \overline{P_{i-m+1}P_{i-m+3}}, \dots, \overline{P_{i+m-3}P_{i+m-1}}, \overline{P_{i+m-1}P_{i+m+1}},$$

(we consider the subscripts as defined modulo $2n$). Thus, if m is odd, then $\mathbf{Q} = (Q_2, Q_4, \dots, Q_{2n})$, and $\mathbf{R} = (R_1, R_3, \dots, R_{2n-1})$, and if m is even, then $\mathbf{Q} = (Q_1, Q_3, \dots, Q_{2n-1})$, and $\mathbf{R} = (R_2, R_4, \dots, R_{2n})$.

The point Q_i may be also described as the point of intersection of the perpendicular bisector planes of the sides

$$\overline{P_{i-m}P_{i-m+2}}, \overline{P_{i-m+2}P_{i-m+4}}, \dots, \overline{P_{i+m-4}P_{i+m-2}}, \overline{P_{i+m-2}P_{i+m}}$$

of \mathbf{P} , and the point R_i may be described as the point of intersection of the bisectorial planes of the angles

$$P_{i-m-1}\widehat{P_{i-m+1}P_{i-m+3}}, P_{i-m+1}\widehat{P_{i-m+3}P_{i-m+5}}, \dots, P_{i+m-3}\widehat{P_{i+m-1}P_{i+m+1}}$$

of \mathbf{P} . Since every angle has two different bisectorial planes, an n -gon has, in general, a large amount of different \mathcal{A} -evolutes.

Proposition 2.1 *The sides of the second \mathcal{P} -evolute of a polygon \mathbf{P} are parallel to the sides of the polygon \mathbf{P} .*

Proof. Let $\mathbf{S} = (S_1, S_3, \dots, S_{2n-1})$ be the \mathcal{P} -evolute of \mathbf{Q} . The points $Q_{i-m+1}, Q_{i-m+3}, \dots, Q_{i+m-3}, Q_{i+m-1}$ belong to the perpendicular bisector hyperplane of the segment $\overline{P_{i-1}P_{i+1}}$. Likewise, the points S_{i-1} and S_{i+1} lie in the perpendicular bisector planes of the segments

$$\overline{Q_{i-m+1}Q_{i-m+3}}, \overline{Q_{i-m+3}Q_{i-m+5}}, \dots, \overline{Q_{i+m-5}Q_{i+m-3}}, \overline{Q_{i+m-3}Q_{i+m-1}}.$$

It follows that the side $\overline{P_{i-1}P_{i+1}}$ of \mathbf{P} and the side $\overline{S_{i-1}S_{i+1}}$ of \mathbf{S} are both perpendicular to the hyperplane spanned by the points $Q_{i-m+1}, \dots, Q_{i+m-1}$; hence they are parallel to each other. (The m points in question do not belong to an $(m-2)$ -dimensional plane.) \square

2.2 Involutives: existence and uniqueness

If an n -gon \mathbf{Q} is the \mathcal{P} -evolute of an n -gon \mathbf{P} , then we call \mathbf{P} a \mathcal{P} -involute of \mathbf{Q} . If an n -gon \mathbf{Q} is one of \mathcal{A} -evolutes of an n -gon \mathbf{P} , then we call \mathbf{P} an \mathcal{A} -involute of \mathbf{Q} .

Lemma 2.2 *The table below contains a complete information on the existence and uniqueness of fixed points and invariant lines for a generic¹ isometry $\sigma: \mathbb{R}^m \rightarrow \mathbb{R}^m$:*

¹The word *generic* means that the statement holds within a dense open set of isometries.

| | m is odd | m is even |
|--------------------------------|---|--|
| σ preserves orientation | no fixed points a unique invariant line | a unique fixed point no invariant lines |
| σ reverses orientation | a unique fixed point a unique invariant line | no fixed points a unique invariant line |

Proof. The isometry σ is a composition of an orthogonal transformation σ_0 and a parallel translation.

Let m be even and σ be orientation preserving. Generically, ± 1 is not an eigenvalue of σ_0 . This means that σ has no invariant lines (even no invariant directions). Let us prove, by induction, that if 1 is not an eigenvalue of σ_0 , then σ has a unique fixed point (for $m = 2$ the fact is obvious).

Let M_0 be an invariant 2-dimensional plane of σ_0 such that the action of σ_0 in M_0 is orientation preserving (this must exist). Then every 2-dimensional plane parallel to M_0 is mapped by σ into a plane parallel to M_0 , and σ is factorized to an isometry $\tilde{\sigma}$ of \mathbb{R}^m/M_0 . The orthogonal transformation $\tilde{\sigma}_0$ also has no eigenvalues 1, so, by the induction hypothesis, $\tilde{\sigma}$ has a unique fixed point. Hence, there exists a unique σ -invariant plane $M \subset \mathbb{R}^m$ parallel to M_0 , and every fixed point of σ must be contained in M . The restriction of σ to M is a rotation by a non-zero angle, it has a unique fixed point, which is a unique fixed point of σ .

If m is odd and σ preserves orientation, then 1 is an eigenvalue of σ_0 , generically, of multiplicity 1. Hence, there exists a line L_0 which is pointwise fixed by σ_0 . By the previous case, the arising isometry $\tilde{\sigma}$ of \mathbb{R}^m/L_0 has a unique fixed point, so there exists a unique σ -invariant line $L \subset \mathbb{R}^m$ parallel to L_0 . If σ has a fixed point, then this point must be contained in L , but the restriction of σ to L is a parallel translation, and it is generically fixed point free.

If m is odd and σ reverses orientation, then σ_0 has an eigenvalue -1 and, generically, no eigenvalue 1. Hence, there is a σ_0 -invariant line L_0 , on which σ_0 acts as a reflection in 0. The arising isometry σ_0 of \mathbb{R}^m/L_0 preserves orientation and 1 is not an eigenvalue of $\tilde{\sigma}_0$; hence, $\tilde{\sigma}$ has a unique fixed point. Thus there is a unique σ -invariant line $L \subset \mathbb{R}^m$, and the direction of L is reversed by σ . Hence, L contains a unique fixed point of σ .

Finally, if m is even and σ reverses orientation, then both 1 and -1

are eigenvalues of σ_0 , generically, both of multiplicity 1. Then there is a σ_0 -invariant plane $M_0 \subset \mathbb{R}^m$, and the restriction of σ_0 to this plane is a reflection in a line. The arising isometry $\tilde{\sigma}$ of \mathbb{R}^m/M_0 preserves orientation, so it has a unique fixed point. Thus, there is a unique σ -invariant plane $M \subset \mathbb{R}^m$, and the restriction of σ to this plane reverses orientation. Generically, it is a glide reflection. Hence, M contains a unique σ -invariant line and no fixed points. \square

Using the lemma, we address the question of the existence and uniqueness of involutes.

Proposition 2.3 *If $n - m$ is odd, then a generic n -gon has no \mathcal{P} -involutes. If $n - m$ is even, then a generic n -gon has a unique \mathcal{P} -involute. For any $n \geq m + 2$, a generic n -gon has a unique \mathcal{A} -involute.*

Proof. In this proof we assume that m is odd; the proof for even m is the same, up to some modification of notations (for example, the points P_i will be numerated by even numbers, while the reflections s_i will be numerated by odd numbers).

Let $\mathbf{Q} = (Q_2, Q_4, \dots, Q_{2n})$ be an n -gon satisfying the conditions formulated at the beginning of this section. Let $s_i, i = 2, 4, \dots, 2n$, be the reflection of \mathbb{R}^m in the $(m - 1)$ -dimensional plane containing the m points $Q_{i-m+1}, Q_{i-m+3}, \dots, Q_{i+m-3}, Q_{i+m-1}$, and let $\sigma = s_{2n} \circ \dots \circ s_4 \circ s_2$. Obviously, for a generic n -gon, σ is a generic isometry, preserving or reversing orientation, depending on the parity of n .

The polygon \mathbf{Q} is a \mathcal{P} -evolute of a polygon $\mathbf{P} = (P_1, P_3, \dots, P_{2n-1})$ if and only if, for every $i = 2, 4, \dots, 2n$, the point Q_i belongs to the perpendicular bisector hyperplanes of the segments

$$\overline{P_{i-m+1}P_{i-m+3}}, \overline{P_{i-m+3}P_{i-m+5}}, \dots, \overline{P_{i+m-5}P_{i+m-3}}, \overline{P_{i+m-3}P_{i+m-1}}.$$

In other words, for every i , the points $Q_{i-m+1}, Q_{i-m+3}, \dots, Q_{i+m-3}, Q_{i+m-1}$ lie in the perpendicular bisector plane of $P_{i-1}P_{i+1}$, which means, in turn, that $P_{i+1} = s_i(P_{i-1})$. Thus, $\sigma(P_1) = P_{2n+1} = P_1$.

If n is even, then σ preserves orientation, and by Lemma 2.2, it has no fixed points, so \mathbf{P} cannot exist. If n is odd, then σ has a generically unique fixed point. We take this point for P_1 and put $P_3 = s_2(P_1)$, $P_5 = s_4(P_3)$, $P_7 = s_6(P_5)$, and so on; we obtain an n -gon \mathbf{P} , whose \mathcal{P} -evolute is \mathbf{Q} .

The proof for \mathcal{A} -involutes is similar, the only difference is that we begin not with a fixed point, but with an invariant line of σ . We successively apply to this line the reflections s_2, s_4, s_6, \dots . We obtain n lines which form an n -gon whose \mathcal{A} -evolute is \mathbf{Q} . (Notice that if we reflect a line in a hyperplane not parallel to this line, then the line and its reflection intersect at the intersection point of the line with the hyperplane.) \square

In the rest of the paper, we will not consider either \mathcal{A} -evolutes or \mathcal{A} -involutes. So, we will refer to \mathcal{P} -evolutes and \mathcal{P} -involutes simply as to evolutes and involutes.

2.3 Spherical polygons and polygons with fixed directions of the sides

A spherical n -gon is a cyclically ordered collection of n points of the unit sphere S^{m-1} ; notation: $\mathbf{v} = (v_2, v_4, \dots, v_{2n})$ or $(v_1, v_3, \dots, v_{2n-1})$. We assume that the spherical polygons are generic: no (cyclically) consecutive m -tuple of vectors v_i is linearly dependent.

The *signature* of a polygon \mathbf{v} is the cyclic sequence \mathbf{s} of signs s_i of the determinants $d_i = \det(v_{i-m+1}, v_{i-m+3}, \dots, v_{i+m-3}, v_{i+m-1})$.

For a spherical n -gon $\mathbf{v} = (v_2, v_4, \dots, v_{2n})$, let $\mathbf{u} = \mathbf{v}^*$ be the *dual* n -gon $\mathbf{u} = (u_1, u_3, \dots, u_{2n-1})$ ($\mathbf{u} = (u_2, u_4, \dots, u_{2n})$, if m is even) where u_i is the positive unit normal vector of the $(m-1)$ -dimensional plane spanned by the vectors $v_{i-m+2}, v_{i-m+4}, \dots, v_{i+m-4}, v_{i+m-2}$. The positivity means that the sign of $u_i \cdot v_{i+m}$ is s_{i+1} , and the sign of $u_i \cdot v_{i-m}$ is $(-1)^{m-1}s_{i-1}$; in particular, both these dot products are not zero.

Lemma 2.4 *If \mathbf{v} is generic, then $\mathbf{u} = \mathbf{v}^*$ is generic.*

Proof. Suppose that

$$A_{i-m+1}u_{i-m+1} + A_{i-m+3}u_{i-m+3} + \dots + A_{i+m-1}u_{i+m-1} = 0. \quad (1)$$

Dot (1) with v_{i+1} ; we get $A_{i-m+1}u_{i-m+1} \cdot v_{i+1} = 0$, and since $u_{i-m+1} \cdot v_{i+1} \neq 0$, we have $A_{i-m+1} = 0$. Taking this into account, dot (1) with v_{i+3} ; we get $A_{i-m+3}u_{i-m+3} \cdot v_{i+3} = 0$, and since $u_{i-m+3} \cdot v_{i+3} \neq 0$, we have $A_{i-m+3} = 0$. And so on. \square

Let $\mathbf{w} = (w_2, w_4, \dots, w_{2n})$ be $(\mathbf{v}^*)^*$.

Lemma 2.5 *One has $w_i = \pm v_i$.*

Proof. Indeed, both v_i and w_i are orthogonal to $m-1$ linearly independent vectors $v_{i-m+2}, v_{i-m+4}, \dots, v_{i+m-4}, v_{i+m-2}$. \square

The following two lemmas provide clarification to Lemmas 2.4 and 2.5. We will not use these statements, and we leave their proofs, which may be regarded as exercises in linear algebra, to the reader.

Lemma 2.6 *One has $w_i = (-1)^{m-1} s_{i-m+3} s_{i-m+5} \dots s_{i+m-5} s_{i+m-3} v_i$.* \square

Let $\{s_i^*\}$ be the signature of the dual polygon \mathbf{u} .

Lemma 2.7 *One has $s_i^* = s_{i-m+2} s_{i-m+4} \dots s_{i+m-4} s_{i+m-2}$.* \square

Fix a spherical n -gon \mathbf{v} , and let $\mathcal{P}_{\mathbf{v}}$ be the space of n -gons in \mathbb{R}^m , considered up to a parallel translation, whose sides are parallel to the respective vectors v_i . If $\mathbf{P} = (P_1, P_3, \dots, P_{2n-1})$ is such a polygon, define the real numbers x_i (signed side lengths) by

$$P_{i+1} - P_{i-1} = x_i v_i.$$

The vector $\mathbf{x} = (x_2, \dots, x_{2n})$ uniquely determines the polygon (up to parallel translation).

Notice that some (or even all) signed side lengths x_i may be zero. In other words, we do not exclude the possibility that some sides collapse to points, but still, when we consider a polygon as belonging to $\mathcal{P}_{\mathbf{v}}$, we assign directions to all the sides.

The coordinates x_i satisfy m linear relations $\sum x_i v_i = 0$, saying that the respective polygon closes up. Thus $\mathcal{P}_{\mathbf{v}}$ is a vector space of dimension $n - m$. Notice that the space $\mathcal{P}_{\mathbf{v}}$ remains the same if we replace some vectors v_i by the opposite vectors (although some coordinates x_i change signs).

2.4 The evolute transformation

In this section we show that the evolute of a polygon $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$ belongs to $\mathcal{P}_{\mathbf{u}}$, where $\mathbf{u} = \mathbf{v}^*$, and that the evolute map $\mathcal{E}: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ is a linear transformation.

Lemma 2.8 *Let $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$ and $\mathbf{Q} = \mathcal{E}(\mathbf{P})$. Then $\mathbf{Q} \in \mathcal{P}_{\mathbf{u}}$, where $\mathbf{u} = \mathbf{v}^*$.*

Proof. The point Q_{i-1} have equal distances to the points

$$P_{i-m-1}, P_{i-m+1}, \dots, P_{i+m-3}, P_{i+m-1},$$

and the point Q_{i+1} have equal distances to the points

$$P_{i-m+1}, P_{i-m+3}, \dots, P_{i+m-1}, P_{i+m+1}.$$

Hence, both these points have equal distances to the points

$$P_{i-m+1}, P_{i-m+3}, \dots, P_{i+m-3}, P_{i+m-1},$$

and the vector $Q_{i+1} - Q_{i-1}$ is orthogonal to the vectors $P_{i-m+3} - P_{i-m+1}, \dots, P_{i+m-1} - P_{i+m-3}$. This means that if $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$, then $\mathbf{Q} \in \mathcal{P}_{\mathbf{v}^*}$. \square

Let a_1, \dots, a_{m+1} be $m+1$ points in \mathbb{R}^m , and let z be their circumcenter.

Lemma 2.9 *The point z is determined by the system of linear equations*

$$z \cdot (a_{i+1} - a_1) = \frac{a_{i+1} \cdot a_{i+1} - a_i \cdot a_i}{2}, i = 1, \dots, m.$$

Proof. The circumcenter z satisfies the equations

$$|z - a_1|^2 = \dots = |z - a_{m+1}|^2,$$

that are equivalent to the above system of linear equations. \square

Let $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$, and let $\mathbf{Q} = \mathcal{E}(\mathbf{P}) \in \mathcal{P}_{\mathbf{u}}$. The vertices P_j of \mathbf{P} and Q_i of \mathbf{Q} are labeled by odd or even integers depending on the parity of m . Define the real numbers (signed side lengths) y_i by

$$Q_{i+1} - Q_{i-1} = y_i u_i.$$

We obtain a vector $\mathbf{y} = \{y_i\}$, and the evolute transformation $\mathcal{E}: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ acts by $\mathbf{x} \mapsto \mathbf{y}$.

Theorem 1 *The vector \mathbf{y} is obtained from \mathbf{x} by a linear transformation depending on \mathbf{v} .*

Proof. Consider $m + 2$ consecutive vertices $P_{i-m-1}, P_{i-m+1}, \dots, P_{i+m-1}, P_{i+m+1}$ of \mathbf{P} ; the $m + 1$ consecutive vectors of the sides are

$$x_{i-m}v_{i-m}, x_{i-m+2}v_{i-m+2}, \dots, x_{i+m-2}v_{i+m-2}, x_{i+m}v_{i+m}.$$

Assume that the first vertex is the origin. Then the $m + 2$ vertices are

$$P_{i-m+2j-1} = \sum_{k=0}^{j-1} x_{i-m+2k}v_{i-m+2k}, \quad j = 0, \dots, m+1. \quad (2)$$

The two relevant vertices of the evolute are Q_{i-1} and Q_{i+1} , and the relevant unit vector is u_i . Recall that Q_{i-1} is the circumcenter of the $m + 1$ points $P_{i-m+2j-1}$, $j = 0, \dots, m$, and Q_{i+1} is the circumcenter of the $m + 1$ points $P_{i-m+2j-1}$, $j = 1, \dots, m+1$.

Apply Lemma 2.9 to $Q_{i-1} \cdot (P_{i-m+2j+1} - P_{i-m+2j-1})$, $j = 0, \dots, m$, to obtain

$$Q_{i-1} \cdot (x_{i-m+2j}v_{i-m+2j}) = \frac{|P_{i-m+2j+1}|^2 - |P_{i-m+2j-1}|^2}{2},$$

which shows, in virtue of (2), that

$$Q_{i-1} \cdot v_{i-m+2j} = \frac{2(x_{i-m}v_{i-m} + \dots + x_{i-m+2j-2}v_{i-m+2j-1}) \cdot v_{i-m+2j} + x_{i-m+2j}}{2}.$$

We see that Q_{i-1} is a linear function of x_{i-m+2j} , $j = 0, \dots, m$; similarly, Q_{i+1} is a linear function of x_{i-m+2j} , $j = 1, \dots, m+1$. Thus, $Q_{i+1} - Q_{i-1}$ is a linear function of x_{i-m+2j} , $j = 0, \dots, m+1$, and so is $y_i = (Q_{i+1} - Q_{i-1}) \cdot u_i$. \square

2.5 Rank of the evolute transformation

The following proposition describes the rank of the evolute transformation.

Proposition 2.10 *Generically, the evolute transformation $\mathcal{E}: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ has 1-dimensional kernel if $n - m$ is odd, and has full rank if $n - m$ is even.*

Proof. If $n - m$ is even, then the map $\mathcal{E}: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ has a full rank, because it is onto: in this case, a generic n -gon has an involute (Proposition 2.3).

Let $n - m$ be odd. We will show that $\dim \text{Ker } \mathcal{E} \geq 1$ and, generically, this dimension is 1. A polygon belongs to the kernel of the evolute map if and only if it is inscribed into a sphere. We will show that, for every \mathbf{v} , the space $\mathcal{P}_{\mathbf{v}}$ contains a polygon inscribed into the unit sphere S^{m-1} and, generically, this polygon is unique up to the reflection in the center.

Take a point $P_1 \in S^{m-1}$ and reflect it, successively, in the hyperplanes passing through the origin and perpendicular to the vectors v_2, v_4, \dots, v_{2n} ; we obtain points $P_3, P_5, \dots, P_{2n+1} \in S^{m-1}$. If $P_{2n+1} = P_1$, then $\mathbf{P} = (P_1, P_3, \dots, P_{2n-1}) \in \mathcal{P}_{\mathbf{v}}$ is inscribed in S^{m-1} , and all polygons from $\mathcal{P}_{\mathbf{v}}$ inscribed in S^{m-1} are obtained by this construction.

The transformation $P_1 \mapsto P_{2n+1}$ is an isometry of S^{m-1} of degree $(-1)^n$. The Lefschetz number of this map is 2, so it has fixed points. Generically, there are precisely 2 fixed points, and they are antipodal. Thus, the space of inscribed n -gons is not zero, and generically has dimension 1. \square

2.6 $\mathcal{P}_{\mathbf{v}}$ as the dual space to $\mathcal{P}_{\mathbf{v}^*}$

Consider an n -gon $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$, where $\mathbf{v} = \{v_i\} \subset S^{m-1}$ is a generic spherical n -gon. Choose an origin and define the support number λ_i as the signed distance from the origin to the $(m-1)$ -dimensional plane which contains the points

$$P_{i-m+1}, P_{i-m+3}, \dots, P_{i+m-3}, P_{i+m-1},$$

parallel to the vectors $v_{i-m+2}, v_{i-m+4}, \dots, v_{i+m-4}, v_{i+m-2}$ and oriented by these vectors. (The polygon \mathbf{P} is not supposed to be generic; for example, some or all its vertices may coincide.)

Let $\mathbf{Q} \in \mathcal{P}_{\mathbf{v}^*}$. Define $\langle \mathbf{P}, \mathbf{Q} \rangle = \sum_i y_i \lambda_i$ where y_i is the signed length of the i -th side of \mathbf{Q} .

Lemma 2.11 *The pairing $\langle \mathbf{P}, \mathbf{Q} \rangle$ is well defined, that is, it does not depend on the choice of the origin.*

Proof. The orienting normal vector of the hyperplane parallel to the vectors $v_{i-m+2}, v_{i-m+4}, \dots, v_{i+m-4}, v_{i+m-2}$ is u_i , therefore $\lambda_i = P_{i+k} \cdot u_i$ for all

$k = -m+1, -m+3, \dots, m-3, m-1$. It follows that $y_i \lambda_i = P_{i+k} \cdot (Q_{i+1} - Q_{i-1})$ for the same values of k , and hence

$$\langle \mathbf{P}, \mathbf{Q} \rangle = \sum_i P_{i+k} \cdot (Q_{i+1} - Q_{i-1}).$$

A parallel translation through vector R changes this expression by

$$\sum_i R \cdot (Q_{i+1} - Q_{i-1}) = R \cdot \sum_i (Q_{i+1} - Q_{i-1}) = 0,$$

as claimed. \square

The pairing $\langle \rangle: \mathcal{P}_{\mathbf{v}} \otimes \mathcal{P}_{\mathbf{v}^*} \rightarrow \mathbb{R}$ is skew-symmetric in the following sense.

Proposition 2.12 *For any $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$, $\mathbf{Q} \in \mathcal{P}_{\mathbf{v}^*}$, one has $\langle \mathbf{P}, \mathbf{Q} \rangle = -\langle \mathbf{Q}, \mathbf{P} \rangle$.*

Proof. If m is odd, then

$$\langle \mathbf{P}, \mathbf{Q} \rangle = \sum_i P_i \cdot (Q_{i+1} - Q_{i-1}) = \sum_i Q_i \cdot (P_{i-1} - P_{i+1}) = -\langle \mathbf{Q}, \mathbf{P} \rangle,$$

and if m is even, then

$$\langle \mathbf{P}, \mathbf{Q} \rangle = \sum_i P_{i+1} \cdot (Q_{i+1} - Q_{i-1}) = \sum_i Q_{i-1} \cdot (P_{i-1} - P_{i+1}) = -\langle \mathbf{Q}, \mathbf{P} \rangle,$$

as claimed. \square

Proposition 2.13 *The pairing $\langle \rangle: \mathcal{P}_{\mathbf{v}} \otimes \mathcal{P}_{\mathbf{v}^*} \rightarrow \mathbb{R}$ is non-degenerate. Thus, with respect to this pairing, $\mathcal{P}_{\mathbf{v}^*}$ is the dual space $(\mathcal{P}_{\mathbf{v}})^*$.*

Proof. We want to prove that for every non-zero set $\{\lambda_i\}$, there exists a set $\{y_i\}$, satisfying the condition $\sum_i y_i u_i = 0$, such that $\sum_i y_i \lambda_i \neq 0$.

The opposite would mean that $\sum_i y_i u_i = 0$ implies $\sum_i y_i \lambda_i = 0$, that is, λ_i is a linear combination of the coordinates of u_i with coefficients not depending on i , $\lambda_i = C \cdot u_i$ for some constant vector C . But $\lambda_i = P_{i+k} \cdot u_i$, for $k = -m+1, -m+3, \dots, m-3, m-1$. Thus $(P_{i+k} - C) \cdot u_i = 0$ for the same values of k .

By applying a parallel translation, we may assume that $C = 0$. The orthogonal complement of the vector u_i is spanned by the vectors $P_{i-m+3} - P_{i-m+1}, P_{i-m+5} - P_{i-m+3}, \dots, P_{i+m-1} - P_{i+m-3}$, implying a linear relation between the m vectors $P_{i+k}, k = -m+1, -m+3, \dots, m-3, m-1$. This holds for every i , hence the polygon \mathbf{P} lies in a hyperplane, contradicting the genericity of \mathbf{v} . \square

The evolute transformation $\mathcal{E} : \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}^*} = (\mathcal{P}_{\mathbf{v}})^*$ is anti-self-adjoint.

Proposition 2.14 *For every $\mathbf{P} \in \mathcal{P}_{\mathbf{v}^*}$, one has $\langle \mathbf{P}, \mathcal{E}(\mathbf{P}) \rangle = 0$.*

Proof. Let $\mathbf{Q} = \mathcal{E}(\mathbf{P})$. If m is odd, then $\langle \mathbf{Q}, \mathbf{P} \rangle = \sum_i Q_i \cdot (P_{i+1} - P_{i-1})$, and according to Lemma 2.9, the last expression equals $\sum_i \frac{|P_{i+1}|^2 - |P_{i-1}|^2}{2} = 0$. For m even, the proof is the same, only instead of Q_i one takes Q_{i+1} . \square

2.7 The cases of “small-gons”: first, second and third evolutes of $(m+2)$ - and $(m+3)$ -gons in \mathbb{R}^m

Since the assumption $n \geq m+2$ was made in the very beginning of Section 2, we start with the case $n = m+2$.

The following statement was proved by E. Tsukerman [11].

Theorem 2 *For a generic $(m+2)$ -gon \mathbf{P} in \mathbb{R}^m , its second evolute $\mathcal{E}^2(\mathbf{P})$ is homothetic to \mathbf{P} .*

(Actually, Tsukerman’s work contains similar results for polygons in all spaces of constant curvature; we restrict ourselves here to the Euclidean case.) We will show that Theorem 2 is an almost immediate corollary of Proposition 2.14.

Proof of Theorem 2. Let \mathbf{P} be a non-zero element of some $\mathcal{P}_{\mathbf{v}}$. Notice that $\dim \mathcal{P}_{\mathbf{v}} = 2$. By Proposition 2.10, the transformation $\mathcal{E} : \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}^*}$ has full rank, so $\mathcal{E}(\mathbf{P}) \neq 0$. By Propositions 2.12, 2.13, and 2.14, both \mathbf{P} and $\mathcal{E}^2(\mathbf{P})$ belong to the orthogonal complement of $\mathcal{E}(\mathbf{P}) \subset \mathcal{P}_{\mathbf{v}^*}$, and this orthogonal complement is 1-dimensional. Thus, \mathbf{P} and $\mathcal{E}^2(\mathbf{P})$ are proportional, that is, the polygons \mathbf{P} and $\mathcal{E}^2(\mathbf{P})$ are homothetic. \square

Now, let us turn to $(m+3)$ -gons.

Theorem 3 *For a generic $(m + 3)$ -gon \mathbf{P} in \mathbb{R}^m , its third evolute $\mathcal{E}^2(\mathbf{P})$ is homothetic to its first evolute $\mathcal{E}(\mathbf{P})$.*

(For $n = 2$, this is Proposition 4.15 of [1].)

Proof. Let \mathbf{P} be a non-zero element of some $\mathcal{P}_{\mathbf{v}}$. By Proposition 2.10, the evolute transformation $\mathcal{E}: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}^*}$ has a 1-dimensional kernel \mathcal{K} . The orthogonal complement $\mathcal{K}^\perp \subset \mathcal{P}_{\mathbf{v}^*}$ is the image of the evolute transformation. We may assume that $\mathcal{E}^2(\mathbf{P}) \notin \mathcal{K}$ (otherwise, $\mathcal{E}^3(\mathbf{P}) = 0$, and the statement of the theorem is trivial). Thus, both $\mathcal{E}(\mathbf{P})$ and $\mathcal{E}^3(\mathbf{P})$ belong to two different two-dimensional subspaces \mathcal{K}^\perp and $\mathcal{E}^2(\mathbf{P})^\perp$ of the three-dimensional space $\mathcal{P}_{\mathbf{v}^*}$. The intersection of these subspaces is one-dimensional, hence the polygons $\mathcal{E}(\mathbf{P})$ and $\mathcal{E}^3(\mathbf{P})$ are homothetic. \square

2.8 Iteration of the evolute transformations on general polygons

The second evolute transformation $\mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ is linear, and it is a standard fact from linear algebra that the asymptotic behavior of the sequence of iterations of this map depends largely on the maximal module eigenvalue of this transformation. If this eigenvalue is real, then sufficiently distant members of this sequence will be almost homothetic to each other.

To make the described property of evolutes better visible, we apply to each member of the sequence two transformations; first, a parallel translation which carries the centroid of the vertices to the origin, and a homothety centered at the origin, which makes the maximal distance from the origin to the vertices equal to 1.

The appearance of the resulting sequence of polygons depends on the maximal eigenvalue of the transformation \mathcal{E}^2 .

If this eigenvalue is real and positive, then, starting with some moment, the polygons will be almost undistinguishable from each other. This case is presented in Figures 1 and 2.

If the maximum module eigenvalue is real and negative, then the shapes of the distant members of the sequence will be almost the same, but the transition from a polygon to the next one will include a flip: the reflection in the origin (this case is presented in Figure 3).

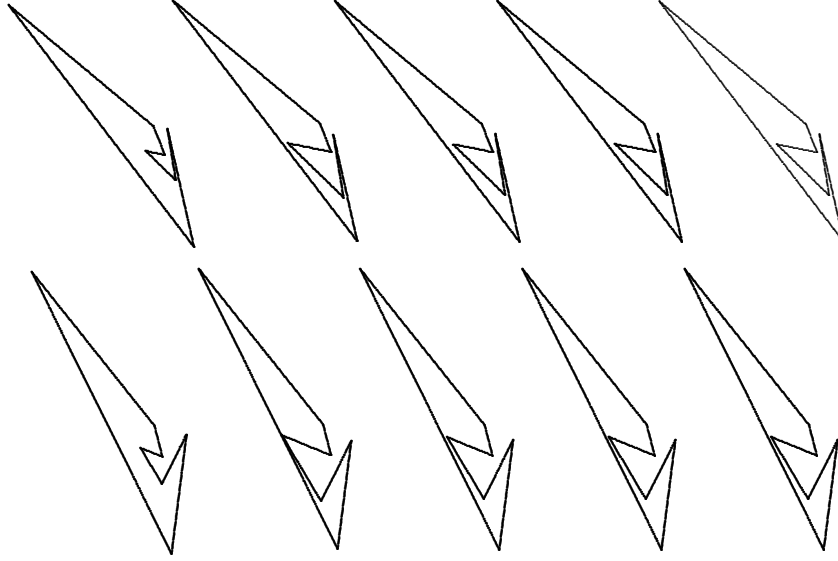


Figure 1: Even-numbered evolutes of a heptagon in \mathbb{R}^3 , two projections.

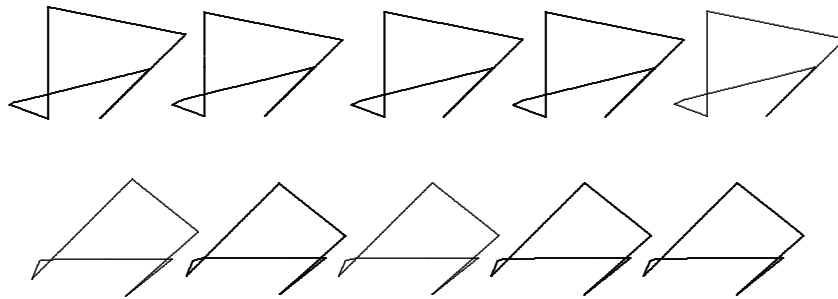


Figure 2: Odd-numbered evolutes of the same heptagon in \mathbb{R}^3 , two projections.

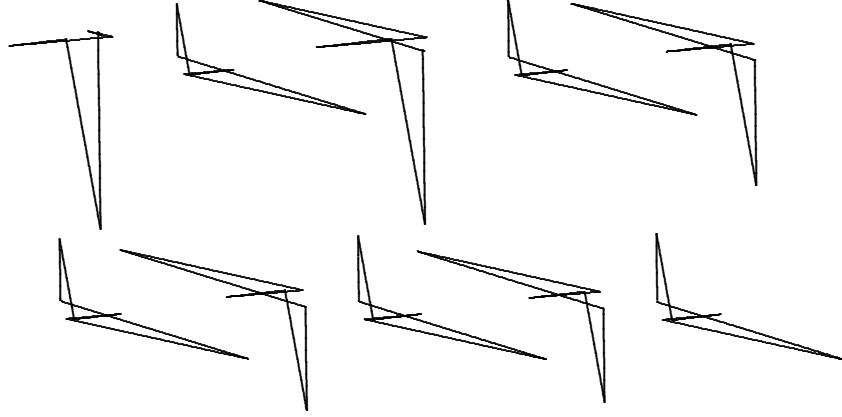


Figure 3: The sequence of even-numbered evolutes of a heptagon in the case when the maximum module eigenvalue of the transformation \mathcal{E}^2 is negative (one projection).

Finally, if the maximum module eigenvalue is not real, then the sequence of evolutes may not reveal periodicity (generically, one observes a quasi-periodic behavior).

The second evolute transformation acts separately on even numbered and odd numbered evolutes of a polygon. Let \mathbf{v} be a spherical polygon and \mathbf{u} its dual.

Proposition 2.15 *The second evolute maps $\mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ and $\mathcal{P}_{\mathbf{u}} \rightarrow \mathcal{P}_{\mathbf{u}}$ have the same eigenvalues.*

Proof. One has two first evolute maps $\mathcal{E}_1: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{u}}$ and $\mathcal{E}_2: \mathcal{P}_{\mathbf{u}} \rightarrow \mathcal{P}_{\mathbf{v}}$. Hence one also has two second evolute maps $\mathcal{E}_2 \circ \mathcal{E}_1: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ and $\mathcal{E}_1 \circ \mathcal{E}_2: \mathcal{P}_{\mathbf{u}} \rightarrow \mathcal{P}_{\mathbf{u}}$, and these two compositions share the eigenvalues. \square

2.9 The spectrum of the second evolute transformation

Section 2.8 may create an impression that, for a generic spherical polygon \mathbf{v} , in the case when the maximum module eigenvalue of the transformation $\mathcal{E}^2: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ is real, the even-numbered evolutes of polygons $\mathbf{P} \in \mathcal{P}_{\mathbf{v}}$ have a prescribed limit shape, that is, there exists a special polygon in $\mathcal{P}_{\mathbf{v}}$ such that

the even-numbered evolutes of almost any polygon from $\mathcal{P}_{\mathbf{v}}$ are asymptotically homothetic to this special polygon. In reality, however, this is not the case: the maximum module eigenvalue of the transformation $\mathcal{E}^2: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ never has multiplicity one.

Theorem 4 *Let \mathbf{v} be a generic spherical polygon. Then every non-zero eigenvalue of the transformation $\mathcal{E}^2: \mathcal{P}_{\mathbf{v}} \rightarrow \mathcal{P}_{\mathbf{v}}$ has even multiplicity, generically multiplicity 2. More precisely, the matrix of the restriction of the transformation \mathcal{E}^2 to the image of \mathcal{E} is conjugated to the block diagonal matrix of the form $\begin{bmatrix} A & 0 \\ 0 & A^T \end{bmatrix}$.*

Proof. We have two spaces, $V = \mathcal{P}_{\mathbf{v}}$ and $U = \mathcal{P}_{\mathbf{v}^*}$, and a non-degenerate pairing $\langle \cdot, \cdot \rangle : U \otimes V \rightarrow \mathbb{R}$ between them, so we identify U with V^* .

We also have two evolute maps, $A: V \rightarrow V^*$ and $B: V^* \rightarrow V$, which are both anti-self-adjoint: $\langle Ax, x \rangle = 0$ and $\langle z, Bz \rangle = 0$ for any $x \in V$ and $z \in V^*$. Equivalently, the anti-self-adjointness property of A and B may be expressed by the equalities $\langle Ax, y \rangle = -\langle Ay, x \rangle$ and $\langle z, Bw \rangle = -\langle w, Bz \rangle$ for any $x, y \in V$ and $z, w \in V^*$.

Furthermore, we assume that A and B are isomorphisms; it is so if $\mathcal{P}_{\mathbf{v}}$ is even-dimensional, and in the odd-dimensional case, we have to replace the spaces $\mathcal{P}_{\mathbf{v}}$ and $\mathcal{P}_{\mathbf{v}^*}$ by the images of the evolute maps.

Let us define a linear symplectic structure on V : $\omega(x, y) = \langle Ax, y \rangle$. It is skew-symmetric, because A is anti-self-adjoint, and non-degenerate, because A is an isomorphism.

A linear transformation L of a symplectic space is called skew-Hamiltonian, if $\omega(Lx, y) = \omega(x, Ly)$ for all x, y (compare with a more familiar definition of a Hamiltonian map H , an infinitesimal symplectomorphism, satisfying the condition $\omega(Hx, y) = -\omega(x, Hy)$).

We claim that the map $BA: V \rightarrow V$ is skew-Hamiltonian. Indeed,

$$\omega(x, BAy) = \langle Ax, BAy \rangle = -\langle ABAy, x \rangle = \langle AB Ax, y \rangle = \omega(BAx, y)$$

(the first and last equalities hold by the definition of ω , and the second and third equalities are the anti-self-adjointness properties of A and ABA).

Now the last and the most important step. Theorem 1 of [4] asserts that the matrix of a skew-Hamiltonian transformation is conjugated by a symplectic matrix to a matrix of the form stated in Theorem 4. \square

Remark 2.16 In dimension two, the inscribed polygons \mathbf{v} and its dual \mathbf{v}^* differ by rotation through 90° , and the evolute map depends only on the exterior angles of a polygon. This makes it possible to identify the space $\mathcal{P}_{\mathbf{v}}$ with $\mathcal{P}_{\mathbf{v}^*}$, and the evolute map A with B . Then the matrix of the second evolute map becomes the square of the matrix of the evolute map. Theorem 7 of [1] states that the spectrum of this evolute map is symmetric with respect to the origin and, furthermore, the opposite eigenvalues have the same geometric multiplicity and the same sizes of the Jordan blocks. This implies the 2-dimensional case of the above Theorem 4.

Remark 2.17 Theorem 4 implies Theorems 2 and 3. Indeed, in these two theorems one deals with maps of two-dimensional spaces, and Theorem 4 implies that these maps are diagonal.

3 Curves

In this section, we restrict ourselves to the three-dimensional case. We hope to consider the situation in higher dimensions, as well as in the elliptic and hyperbolic geometries, in the near future.

3.1 Spherical curves

Let us recall some facts about spherical curves; see, e.g., [2].

A coorientation of a smooth curve on the unit sphere is a choice of a unit normal vector field. Let γ be a cooriented closed spherical curve. The dual curve $\bar{\gamma}$ is obtained by moving each point of γ distance $\pi/2$ in the direction of the coorientation.

For example, the Southward cooriented Arctic Circle is dual to the Tropic of Capricorn, and the Northward cooriented Arctic Circle is dual to the Tropic of Cancer.

The dual curve is the locus of centers of great circles tangent to the original curve. The second dual curve is antipodal to the original one.

An inflection of a spherical curve is a point where the geodesic curvature vanishes, that is, where the curve is abnormally well approximated by a great circle. The spherical duality interchanges inflection points with cusps.

We shall be concerned with locally convex, i.e., inflection free, smooth curves. This class of curves is invariant under the duality.

We shall need the following property of the tangent indicatrix of a spherical curve: its total geodesic curvature is zero (equivalently, the appropriately defined, area bounded by the tangent indicatrix of a spherical curve is a multiple of 2π), see [2, 9]. This property extends to spherical curves with cusps.

3.2 Curves in space and their tangent indicatrices

Let $\Gamma(x)$ be an arc length parameterized curve in \mathbb{R}^3 with non-vanishing curvature k and non-vanishing torsion τ . Let (T, N, B) be the Frenet frame, and let $\gamma(t)$ be the arc length parameterized tangent indicatrix. Set $\rho = 1/k$, the radius of curvature.

Lemma 3.1 *One has:*

$$\frac{dx}{dt} = \rho, \quad \frac{dt}{dx} = k.$$

The tangent indicatrix γ is smooth and inflection free. Its spherically dual curve $\bar{\gamma}$ is given by the vector B .

Proof. The Frenet equations read:

$$T_x = kN, N_x = \tau B - kT, B_x = -\tau N.$$

One has:

$$1 = |T_t| = |T_x|x_t = kx_t,$$

hence $x_t = \rho$.

Since $\gamma = T$ and $k \neq 0$, the velocity of γ does not vanish, so γ is smooth. The vector B is orthogonal to T , hence tangent to the sphere, and normal to N , therefore it gives the tangent indicatrix a coorientation and describes the dual curve. Since $\tau \neq 0$, the Frenet equations imply that this dual curve is smooth as well. \square

Remark 3.2 The geodesic curvature vector of the tangent indicatrix is $T + T_{tt}$. A straightforward calculation using the Frenet formulas yields $\rho\tau$ for the geodesic curvature of this spherical curve. At a cusp (at^2, bt^3, ct^4) , both the curvature k and the torsion τ are infinite, but the product $\rho\tau$ has a finite non-zero limit.

Let $\gamma(t)$ be a closed smooth locally convex arc length parameterized spherical curve. Consider the piecewise smooth spatial curves Γ having γ as the tangent indicatrix, modulo parallel translations. The space of such curves is identified with the space of periodic functions $\rho(t)$, satisfying the linear relation

$$\int \rho(t) \gamma(t) dt = 0,$$

saying that Γ is a closed. The zeros of the function ρ correspond to the cusps of the respective space curve Γ . Denote the space of such curves Γ by \mathcal{C}_γ . This space is a continuous analog of the space $\mathcal{P}_\mathbf{v}$.

Let $\bar{\Gamma}$ be the evolute of Γ and y be its arc length parameter. The formula for the evolute, originally due to Monge, can be found in, e.g., [10, 12] and [5]:

$$\bar{\Gamma} = \Gamma + \rho N + \frac{\rho_x}{\tau} B.$$

Furthermore, $\bar{\Gamma}_x = \Phi B$, where

$$\Phi = \rho\tau + \left(\frac{\rho_x}{\tau}\right)_x \quad (3)$$

see [12] and [5] again. Therefore

$$\frac{dy}{dx} = \Phi, \quad \frac{dx}{dy} = \frac{1}{\Phi}. \quad (4)$$

Remark 3.3 At a cusp (at^2, bt^3, ct^4) , the quantity ρ_x/τ has a finite limit, and the center of the osculating sphere is located on the binormal, the vertical axis, at distance $a^2/(2c)$ from the origin.

Since the velocity vector of the evolute $\bar{\Gamma}$ is proportional to the binormal of Γ , we have

Corollary 3.4 *The tangent indicatrix of $\bar{\Gamma}$ is the curve $\bar{\gamma}$ spherically dual to the curve γ , the tangent indicatrix of Γ .*

Remark 3.5 Equation (3) implies that

$$\Phi = \rho\tau + \left(\frac{\rho_x}{\tau}\right)_x = 0$$

is a necessary and sufficient condition for a spatial curve to lie on a sphere. This is a classical result, see, e.g., [10].

Let $\bar{\rho}$ and $\bar{\tau}$ be the radius of curvature and the torsion of $\bar{\Gamma}$.

Lemma 3.6 *One has*

$$\bar{\rho} = -\frac{\Phi}{\tau}, \quad \bar{\tau} = \frac{1}{\Phi}. \quad (5)$$

Proof. Let $(\bar{T}, \bar{N}, \bar{B})$ be the Frenet frame along $\bar{\Gamma}$. Then $\bar{T} = B$ and, using the Frenet formulas,

$$\frac{d\bar{T}}{dy} = \frac{1}{\Phi} \frac{dB}{dx} = -\frac{\tau}{\Phi} N.$$

Hence $\bar{N} = N$ and $\bar{k} = -\frac{\tau}{\Phi}$. This implies the first equation of the lemma.

Also $\bar{B} = \bar{T} \times \bar{N} = B \times N = -T$. Next,

$$\frac{d\bar{B}}{dy} = -\frac{1}{\Phi} \bar{N},$$

therefore $\bar{\tau} = \frac{k}{\Phi}$, implying the second equation. \square

Let s be the arc length parameter on $\bar{\gamma}$. The next result describes the operation of taking the evolute as a linear operator $\mathcal{E} : \mathcal{C}_\gamma \rightarrow \mathcal{C}_{\bar{\gamma}}$; this is a continuous analog of Theorem 1.

Theorem 5 *One has:*

$$\bar{\rho} = -\rho - \rho_{ss}.$$

Proof. We have

$$\frac{dy}{ds} = \bar{\rho}, \quad \frac{ds}{dy} = \bar{k}. \quad (6)$$

Combining (4), (5), and (6), yields

$$\frac{d}{ds} = -\frac{1}{\tau} \frac{d}{dx}.$$

Using (3) and (5), one obtains

$$\bar{\rho} = -\rho - \frac{1}{\tau} \left(\frac{\rho_x}{\tau} \right)_x = -\rho - \rho_{ss},$$

as claimed. \square

3.3 The kernel and the image of the evolute map

This section is a continuous counterpart to Section 2.5.

Let γ be a generic smooth closed locally convex spherical curve. Consider the evolute map $\mathcal{E} : \mathcal{C}_\gamma \rightarrow \mathcal{C}_{\bar{\gamma}}$. The next proposition is an analog of Proposition 2.10: the infinite-dimensional space \mathcal{C}_γ , in a sense, is odd-dimensional.

Proposition 3.7 *The map \mathcal{E} is a linear bijection.*

Proof. Similarly to the polygonal case, $\text{Ker } \mathcal{E}$ consists of the curves in \mathcal{C}_γ that lie on a sphere. As we mentioned earlier, the tangent indicatrix of a spherical curve has zero total geodesic curvature. This condition fails for a generic γ , hence $\text{Ker } \mathcal{E} = 0$.

Given $\bar{\Gamma} \in \mathcal{C}_{\bar{\gamma}}$, we want to construct its involute, a curve Γ whose evolute is $\bar{\Gamma}$. As we mentioned in Introduction, the evolute of a curve is the envelope of its normal planes. That is, Γ is normal to the family of the osculating planes of $\bar{\Gamma}$.

Let y be a parameter on $\bar{\Gamma}$, and consider the family $\xi(y)$ of the osculating planes of $\bar{\Gamma}$. Since $\bar{\gamma}$ is a closed curve, the curve $\bar{\Gamma}$ has an even number of cusps. The Frenet basis (T, N) gives the osculating planes of $\bar{\Gamma}$ consistent orientations. Thus $\xi(y)$ is a loop in the space of oriented planes in \mathbb{R}^3 , and we need to find a closed curve, normal to this 1-parameter family of planes.

Start with some plane $\xi(y_0)$, and pick a point A in this plane. There is a unique curve through A , orthogonal to our family of planes. After one traverses the curve $\bar{\Gamma}$, the orthogonal curve returns to the same plane $\xi(y_0)$, say, at point B . This defines a map $A \mapsto B$ of this plane, and we want to find its fixed point.

We claim that this map of the plane $\xi(y_0)$ is an orientation-preserving isometry. It suffices to establish an infinitesimal version of this claim. Indeed, the map $\xi(y) \rightarrow \xi(y + dy)$, given by the orthogonal curves to the family of planes, is a rotation about the intersection line of these two infinitesimally close planes (this line is tangent to the curve $\bar{\Gamma}$).

Thus we have an orientation preserving isometry of the plane $\xi(y_0)$, generically a rotation. A rotation has a unique fixed point, as needed. \square

Remark 3.8 One can show that the angle of the rotation $\xi(y_0) \rightarrow \xi(y_0)$ equals the total curvature of the curve $\bar{\Gamma}$.

3.4 \mathcal{C}_γ as dual space to $\mathcal{C}_{\bar{\gamma}}$

This section is a continuous counterpart to Section 2.6: for a generic spherical curve γ , we will define a non-degenerate pairing between \mathcal{C}_γ and $\mathcal{C}_{\bar{\gamma}}$ and will prove that the evolute transformation is anti-self-adjoint.

We continue using the same notations: for a $\Gamma \in \mathcal{C}_\gamma$, we denote by ρ and τ the inverse curvature and torsion, by x the arc length parameter, and by T, N, B the Frenet frame; t denotes the arc length parameter on γ . For $\bar{\gamma}$ and $\bar{\Gamma} \in \mathcal{C}_{\bar{\gamma}}$, the notations become $\bar{\rho}, \bar{\tau}, y$, and s , respectively.

The support number $\lambda(x)$ is defined as the signed distance from the origin to the osculating plane of Γ at the point $\Gamma(x)$; in other words, $\lambda(x) = \Gamma(x) \cdot B(x)$. For $\Gamma \in \mathcal{C}_\gamma, \bar{\Gamma} \in \mathcal{C}_{\bar{\gamma}}$, we put

$$\langle \Gamma, \bar{\Gamma} \rangle = \int_{\bar{\gamma}} \lambda(x(s)) \bar{\rho}(s) ds. \quad (7)$$

(Notice that $\rho = \frac{dx}{dt}$ and $\bar{\rho} = \frac{dy}{ds}$ are the analogs of the side lengths for polygons.)

The next proposition is a continuous analog of Lemma 2.11, and Propositions 2.12 and 2.13.

Proposition 3.9 *For a generic γ , the pairing $\langle \cdot \rangle$ is well-defined, anti-symmetric, and non-degenerate.*

Proof. We claim that

$$\int_{\bar{\gamma}} \lambda(x(s)) \bar{\rho}(s) ds = \int_{\bar{\gamma}} \Gamma \cdot \bar{\Gamma}_s ds.$$

Indeed, $\bar{\Gamma}_s = \bar{\Gamma}_y y_s = \bar{\gamma} \bar{\rho} = B \bar{\rho}$, hence $\Gamma \cdot \bar{\Gamma}_s = (\Gamma \cdot B) \bar{\rho} = \lambda \bar{\rho}$.

It follows that a parallel translation through vector R results in the following change of (7):

$$\int_{\bar{\gamma}} R \cdot \bar{\Gamma}_s ds = R \cdot \int_{\bar{\gamma}} \bar{\Gamma}_s ds = 0.$$

Therefore the pairing does not depend on the choice of the origin.

Concerning anti-symmetry,

$$\langle \Gamma, \bar{\Gamma} \rangle = \int \Gamma \cdot \bar{\Gamma}_s ds = \int \Gamma \cdot d\bar{\Gamma} = - \int \bar{\Gamma} \cdot d\Gamma = -\langle \bar{\Gamma}, \Gamma \rangle.$$

Next, we prove non-degeneracy. For $\Gamma \in \mathcal{C}_\gamma$, we want to find $\bar{\Gamma} \in \mathcal{C}_{\bar{\gamma}}$ such that $\langle \Gamma, \bar{\Gamma} \rangle \neq 0$; in other words, we want to find a function $\bar{\rho}(s)$ such that

$$\int_{\bar{\gamma}} \bar{\rho} \bar{\gamma} ds = 0, \text{ but } \int_{\bar{\gamma}} \lambda \bar{\rho} ds \neq 0.$$

It is clear that we can find such $\bar{\rho}$ if and only if the function $\lambda(s)$ is not a linear combination with constant coefficients of the three components of the vector function $\bar{\gamma}$, that is, the binormal vector B .

The function $\Gamma \cdot B$ is a linear combination of the components of the vector B if and only if $\Gamma \cdot B = C \cdot B$ for some constant vector C , that is, when $(\Gamma - C) \cdot B = 0$, or $(\Gamma - C) \cdot (T \times N) = 0$, that is, $\det[\Gamma - C, \Gamma', \Gamma''] = 0$ identically (here prime means d/dx).

We claim that in this case Γ must be planar. Indeed, applying a parallel translation, assume that $C = 0$. Then $\Gamma = f\Gamma' + g\Gamma''$ for some functions f and g , hence $\Gamma' = f'\Gamma' + (f + g')\Gamma'' + g\Gamma'''$ and, using the Frenet formulas,

$$(f' - 1 - gk^2)T + (kf + kg' + k'g)N + gk\tau B = 0.$$

If $\tau = 0$ then the curve is planar, and if $\tau \neq 0$ almost everywhere then $g = 0$, and $\Gamma = f\Gamma'$, which implies that Γ is a line.

Since γ is generic, Γ is not planar, and we are done. \square

Assume now that $\bar{\Gamma}$ is the evolute of Γ . In this case, $\bar{\rho} = -\frac{\Phi}{\tau}$ (Lemma 3.6) and $ds = -\tau dx$ (Theorem 5). The next result is a continuous analog of Proposition 2.14.

Proposition 3.10 *One has $\langle \Gamma, \bar{\Gamma} \rangle = 0$.*

Proof. Using the formula for Φ , the Frenet formulas, and integration by parts, we have

$$\begin{aligned} \int_{\bar{\gamma}} \lambda \bar{\rho} ds &= \int_{\Gamma} (\Gamma \cdot B) \Phi dx = \int_{\Gamma} (\Gamma \cdot B) \rho \tau dx + \int_{\Gamma} (\Gamma \cdot B) \left(\frac{\rho_x}{\tau} \right)_x dx = \int_{\Gamma} \rho (\Gamma \cdot \tau B) dx \\ &- \int_{\Gamma} (\Gamma \cdot B)_x \frac{\rho_x}{\tau} dx = \int_{\Gamma} \rho (\Gamma \cdot (N_x + kT)) dx - \int_{\Gamma} (T \cdot B) \frac{\rho_x}{\tau} dx + \int_{\Gamma} (\Gamma \cdot N) \rho_x dx. \end{aligned}$$

Since $\Gamma \cdot N_x = (\Gamma \cdot N)_x$, $\Gamma \cdot T = \frac{1}{2}(\Gamma \cdot \Gamma)_x$, and $T \cdot B = 0$, the last expression equals

$$\int_{\bar{\gamma}} [\rho (\Gamma \cdot N)]_x dx + \frac{1}{2} \int_{\bar{\gamma}} (\Gamma \cdot \Gamma)_x dx = 0,$$

as claimed. \square

3.5 Spatial hypocycloids

Consider an arc length parameterized circle of latitude

$$x = r \cos \left(\frac{t}{r} \right), \quad y = r \sin \left(\frac{t}{r} \right), \quad z = \sqrt{1 - r^2}, \quad (8)$$

with $0 \leq t \leq 2\pi r$. Let $\rho(t)$ be the respective $2\pi r$ -periodic function. Then

$$\int_0^{2\pi r} \cos \left(\frac{t}{r} \right) \rho(t) dt = \int_0^{2\pi r} \sin \left(\frac{t}{r} \right) \rho(t) dt = \int_0^{2\pi r} \rho(t) dt = 0.$$

Set $\alpha = \frac{t}{r}$; then

$$\int_0^{2\pi} \cos \alpha \rho(\alpha) d\alpha = \int_0^{2\pi} \sin \alpha \rho(\alpha) d\alpha = \int_0^{2\pi} \rho(\alpha) d\alpha = 0,$$

that is, the Fourier expansion of $\rho(\alpha)$ is free from the constant term and from the first harmonics.

We mention, in passing, that this implies that the function $\rho(\alpha)$ has at least four zeros on the interval $[0, 2\pi)$; see, e.g., [6].

Next, consider the arc length parameter s on the dual circle of latitude. One has

$$s = \frac{\sqrt{1 - r^2}}{r} = \sqrt{1 - r^2} \alpha,$$

hence

$$\bar{\rho} = -\rho - \rho_{ss} = -\rho - \frac{1}{1 - r^2} \rho_{\alpha\alpha}. \quad (9)$$

Consider now the case when the function ρ is a pure harmonic: $\rho(\alpha) = \cos k\alpha$ with integral $k \geq 2$. Then, by formula (9), $\bar{\rho}$ is proportional to ρ . In this case, the parametric equations of the curve $\Gamma(t)$ (with the indicatrix being the circle of latitude (8)) are

$$\begin{aligned} x &= r \left(\frac{\sin(k-1)t}{k-1} + \frac{\sin(k+1)t}{k+1} \right), \\ y &= r \left(\frac{\cos(k-1)t}{k-1} - \frac{\cos(k+1)t}{k+1} \right), \quad z = 2\sqrt{1 - r^2} \frac{\sin kt}{k}. \end{aligned} \quad (10)$$

It follows from formula (9) (and can be confirmed by a direct computation) that the evolute $\bar{\Gamma}$ is obtained from Γ by switching $r \longleftrightarrow \sqrt{1-r^2}$ and a homothety with the coefficient $\frac{r^2 + k^2 - 1}{r\sqrt{1-r^2}}$.

Corollary 3.11 *The second evolute $\bar{\bar{\Gamma}}(t)$ is obtained from $\Gamma(t)$ by a homothety with the coefficient $\frac{r^2(1-r^2) + k^2(k^2-1)}{r^2(1-r^2)}$.*

This makes the curve $\Gamma(t)$ similar to the classical hypocycloid.

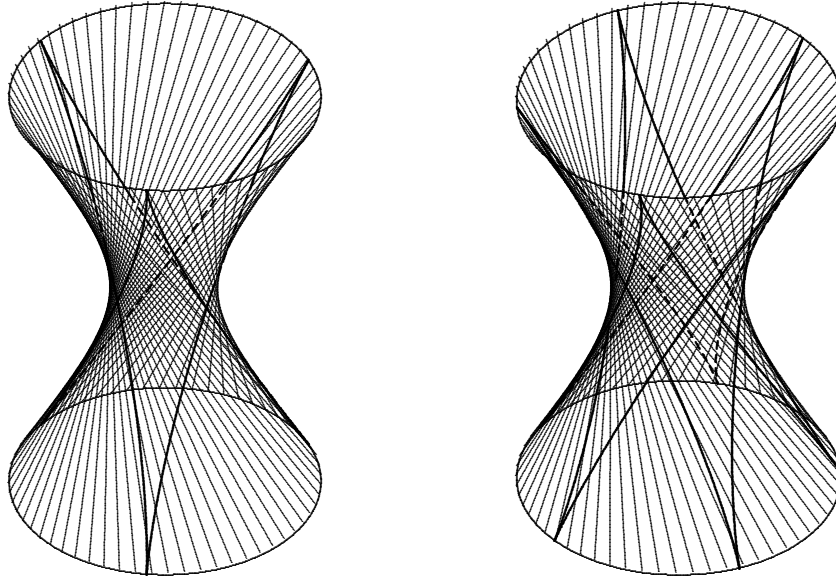


Figure 4: Spatial hypocycloids: $k = 3$ and $k = \frac{5}{2}$

Let us provide some geometric information regarding these “hypocycloids.” The next proposition is proved by a straightforward calculation.

Proposition 3.12 *The curve $\Gamma(t)$ with the parametric equations (10) is contained in the hyperboloid*

$$\frac{k^2 - 1}{4r^2}(x^2 + y^2) - \frac{k^2}{4(1-r^2)}z^2 = \frac{1}{k^2 - 1}$$

between the planes $z = \pm \frac{2\sqrt{1-r^2}}{k}$. It has $2k$ cusps at the points

$$\left(2kr \cos \frac{2i-1}{2k}, 2kr \sin \frac{2i-1}{2k}, 2(-1)^i \frac{\sqrt{1-r^2}}{k} \right), \quad i = 1, 2, \dots, 2k.$$

The number k may be rational, $k = p/q > 1$, $(p, q) = 1$. In this case, the indicatrix will be the circle (8) traversed q times, the curve Γ will be contained in the same hyperboloid and will have $2p$ cusps. See Figure 4.

References

- [1] M. Arnold, D. Fuchs, I. Izmistiev, S. Tabachnikov, E. Tsukerman. *Iterating evolutes and involutes*. Preprint arXiv:1510:07742v2.
- [2] V. Arnold. *The geometry of spherical curves and quaternion algebra*. Russian Math. Surveys **50** (1995), 1–68.
- [3] W. Blaschke, K. Leichtweiß. *Elementare Differentialgeometrie*. Springer-Verlag, Berlin-New York, 1973.
- [4] H. Faßbender, D. S. Mackey, N. Mackey, H. Xu. *Hamiltonian square roots of skew-Hamiltonian matrices*. Lin. Alg. Appl. **287** (1999), 125–159.
- [5] D. Fuchs. *Evolutes and involutes of spatial curves*. Amer. Math. Monthly **120** (2013), 217–231.
- [6] D. Fuchs, S. Tabachnikov. *Mathematical omnibus. Thirty lectures on classic mathematics*. Amer. Math. Soc., Providence, RI, 2007.
- [7] B. Grünbaum. *Quadrangles, pentagons, and computers*. Geombinatorics **3** (1993), no. 1, 4–9.
- [8] B. Grünbaum. *Quadrangles, pentagons, and computers, revisited*. Geombinatorics **4** (1994), no. 1, 11–16.
- [9] B. Solomon. *Tantrices of spherical curves*. Amer. Math. Monthly **103** (1996), 30–39.

- [10] D. Struik. *Lectures on classical differential geometry*. Addison-Wesley Press, Inc., Cambridge, Mass., 1950.
- [11] E. Tsukerman. *The perpendicular bisector construction in n -dimensional Euclidean and non-Euclidean geometries*. Preprint arXiv:1203.6429.
- [12] R. Uribe-Vargas. *On vertices, focal curvatures and differential geometry of space curves*. Bull. Braz. Math. Soc. **36** (2005), 285–307.